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SOLUTION OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we apply the Adomian decomposition method (ADM) for solving nonlinear fractional di£erential equations (FDEs) of Caputo sense. The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed.

Keywords: Fractional di£erential equation; Adomian Method; Existence; Uniqueness; Error analysis.

1. INTRODUCTION

Fractional Differential equations (FDEs) have many applications in engineering and science, including electrical networks, .uid .ow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems ([1]-[13]). In this paper, Adomian decomposition method (ADM) ([14]-[23]) is used to solve nonlinear FDEs of Caputosence. This method has many advantages, it is e¢ ciently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization.

The paper is organized as follows: In section two ADM is applied to the problem under consideration. In section three uniqueness, convergence; error analysis are discussed. Finally two numerical examples are presented by using MATHEMATICA package.

2. FORMULATION OF THE PROBLEM

Consider the nonlinear FDE,

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) + a(t) f(y(t)) = x(t) \tag{1}$$

subject to the initial conditions,

$$y^{(j-1)}(0) = c_j,\ j = 1, 2, \ldots, n. \tag{2}$$

where,

$$_0\mathcal{D}_t^{\sigma_n} \equiv {}_0D_t^{\sigma_n}\ {}_0D_t^{\sigma_{n-1}}\ {}_0D_t^{\sigma_{n-2}} \ldots {}_0D_t^{\sigma_1},$$

$$\sigma_n = \sum_{k=1}^{n} \alpha_k,\ 0 \le \alpha_k \le 1,\ k = 1, 2, \ldots n.$$

the fractional derivative is of sequential Caputo sense. In the applications, the Caputo sense are preferred to use because the initial conditions of $y(t)$ and its derivatives will be of integer orders and has a physical meaning. Now performing subsequently the fractional integration of order α_n, α_{n-1}, ..., α_1, this reduces the problem (1)-(2) to the fractional integral equation (FIE).

$$y(t) = \sum_{j=1}^{n} \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_{j-1}} + \frac{2}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_{n-1}} x(\tau) dr$$
$$- \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_{n-1}} f(y(\tau)) d\tau \tag{3}$$

Assume that $x(t)$ is bounded $\forall\ t \in J = [0, T]$, $T \in R^+$, $|a(\tau)| \le M\ \ \forall\ 0 \le \tau \le t \le T$, M is a finite constant and $f(y)$ is Lipschitz continous with Lipschitz constant L such as,

$$|f(y) - f(z)| \le L|y - z| \tag{4}$$

and has Adomian polynomials representation,

$$f(y) = \sum_{n=0}^{\infty} A_n(y_0, y1, \ldots y_n) \tag{5}$$

where,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f\left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0} \tag{6}$$

substitute from equation (5) into equation (3) we get,

$$y(t) = \sum_{j=1}^{n} \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_{j-1}} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_{n-1}} x(\tau) d\tau$$
$$- \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_{n-1}} \sum_{n=0}^{\infty} A_n d\tau \tag{7}$$

Let $y(t) = \sum_{n=0}^{\infty} y_n(\tau)$ in (7) and applying ADM, we get the following recursive relations,

$$y_0(t) = \sum_{j=1}^{n} \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_j - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} x(\tau) d\tau, \tag{8}$$

$$y_i(t) = -\frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n - 1} A_{i-1} d\tau, \; i \geq 1. \tag{9}$$

Finally, the solution is,

$$y(t) = \sum_{i=0}^{\infty} y_i(t) \tag{10}$$

2.1 Existence and Uniqueness

Theorem 1: If $0 < \alpha < 1$ where $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n + 1)}$, then the series (3.10) is the solution of the problem (1)-(2) and this solution is unique.

Proof: For existence,

$$\begin{aligned}
y(t) &= \sum_{i=0}^{\infty} y_i(t) \\
&= y_0(t) + \sum_{i=0}^{\infty} y_i(t) \\
&= y_0(t) - \frac{1}{\Gamma(\sigma_n)} \sum_{i=0}^{\infty} \int_0^t a(\tau)(t-\tau)^{\sigma_n - 1} A_{i-1} d\tau \\
&= y_0(t) - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n - 1} \sum_{i=1}^{\infty} A_{i-1} d\tau \\
&= y_0(t) - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n - 1} \sum_{i=1}^{\infty} A_i d\tau \\
&= \sum_{j=1}^{n} \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_j - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} x(\tau) d\tau \\
&\quad - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau))(t-\tau)^{\sigma_n - 1} f(y(\tau) d\tau
\end{aligned}$$

then the Adomian's series solution satisfy equation (3) which is the equivalent FIE to the problem (1)-(2).

For uniqueness of the solution

Assume that y and z are two di£erent solutions to the problem (1)-(2) and hence,

$$|y-z| = \left|\frac{1}{\Gamma(\sigma_n)}\int_0^t a(\tau))(t-\tau)^{\sigma_n-1}[f(y)-f(z)]d\tau\right|$$

$$\leq \frac{1}{\Gamma(\sigma_n)}\int_0^t (t-\tau)^{\sigma_n-1}|a(\tau)||f(y)-f(z)|d\tau$$

$$\leq \frac{LM}{\Gamma(\sigma_n)}|y-z|\int_0^t (t-\tau)^{\sigma_n-1}d\tau$$

$$\leq \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}|y-z|$$

Let $\frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)} = \alpha$ where, $0 < \alpha < 1$ then,

$$|y-z| \leq \alpha|y-z|$$

$$(1-\alpha)\,|y-z| \leq 0$$

but, $(1-\alpha)\,|y-z| \geq 0$ and since, $(1-\alpha) \neq 0$ then, $|y-z| = 0$ this imply that, $y = z$ and this completes the proof.

2.2 Proof of Convergence

Theorem 2: The series solution (10) of the problem (1)-(2) using ADM converges if $|y_1| < \infty$ and $0 < \alpha < 1$, $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}$.

Proof: Define the Banach space $(C[J], \|\cdot\|)$, the space of all continuous functions on J with the norm $\|y(t)\| = \max_{t\in J}|y(t)|$. Define the sequence $\{S_n\}$ such that, $S_n = \sum_{i=0}^{n} y_i(t)$ the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} y_i(t)$ since,

$$f(y) = f\left(\sum_{i=0}^{\infty} y_i(t)\right) = \sum_{i=0}^{\infty} A_i(y_0, y_1, \ldots, y_i)$$

so,

$$f(y_0) = f(S_0) = A_0,$$

$$f(y_0 + y_1) = f(S_1) = A_0 + A_1,$$
$$f(y_0 + y_1 + y_2) = f(S_2) = A_0 + A_1 + A_2,$$
$$\vdots$$
$$f(S_n) = \sum_{i=0}^{n} A_i(y_0, y_1, \ldots, y_i).$$

Let, S_n and S_m be two arbitrary partial sums with, $n \geq m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
\|S_n - S_m\| &= \max_{t\in J}\left|S_n - S_m\right| = \max_{t\in J}\left|\sum_{i=m+1}^{n} y_i(t)\right| \\
&= \max_{t\in J}\left|\sum_{i=m+1}^{n} -\frac{1}{\Gamma(\sigma_n)}\int_0^t a(\tau)(t-\tau)^{\sigma_{n-1}} A_{i-1} d\tau\right| \\
&= \max_{t\in J}\left|\frac{1}{\Gamma(\sigma_n)}\int_0^t a(\tau)(t-\tau)^{\sigma_{n-1}} \sum_{i=m}^{n-1} A_i d\tau\right| \\
&= \max_{t\in J}\left|\frac{1}{\Gamma(\sigma_n)}\int_0^t a(\tau)(t-\tau)^{\sigma_{n-1}} [f(S_{n-1}) - f(S_{m-1}) d\tau\right| \\
&\leq \frac{LM}{\Gamma(\sigma_n)} \max_{t\in J}\left|S_{n-1} - S_{m-1}\right| \int_0^t (t-\tau)^{\sigma_{n-1}} d\tau \\
&\leq \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}\|S_{n-1} - S_{m-1}\| \\
&\leq \alpha\|S_{n+1} - S_{m-1}\|
\end{aligned}$$

Let $n = m + 1$ then,

$$\begin{aligned}
\|S_{m+1} - S_m\| &\leq \alpha\|S_m - S_{m-1}\| \\
&\leq \alpha^2\|S_{m-1} - S_{m-2}\| \leq \ldots \leq \alpha^m\|S_1 - S_0\|
\end{aligned}$$

From the triangle inequality we have,

$$\begin{aligned}
\|S_n - S_m\| &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \ldots + \|S_n - S_{n-1}\| \\
&\leq [\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1}\|S_1 - S_0\| \\
&\leq \alpha^m[1 + \alpha + \ldots + \alpha^{n-m-1}\|S_1 - S_0\| \\
&\leq \alpha^m\left[\frac{1-\alpha^{n-m}}{1-\alpha}\right]\|y_1(t)\|
\end{aligned}$$

Since, $0 < \alpha < 1$, and $n \geq m$ then, $(1 - \alpha^{n-m}) \leq 1$. Consequently,

$$\|S_n - S_m\| \leq \frac{1-\alpha^m}{1-\alpha}\|y_1(t)\|$$

$$\leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$$

but, $|y_1(t)| \leq \infty$ and as $m \to \infty$ then, $\|S_n - S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{n=0}^{\infty} y_n(t)$ converges and the proof is complete.

2.3 Error Analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

Theorem 3: The maximum absolute truncation error of the series solution (10) to the problem (1)-(2) is estimated to be,

$$\max_{t\in J}\left|y(t) - \sum_{i=0}^{m} y_i(t)\right| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$$

Proof: From Theorem 2 we have

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$$

but, $S_n = \sum_{i=0}^{n} y_i(t)$ as $n \to \infty$ then, $S_n \to y(t)$ so,

$$\|y(t) - S_m\| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$$

so, the maximum absolute truncation error in the interval J is,

$$\max_{t\in J}\left|y(t) - \sum_{i=0}^{m} y_i(t)\right| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$$

and this completes the proof.

3. NUMERICAL EXAMPLES

Example 1: Consider the initial value problem [20],

$$D^{\mu} y = y^2 + 1,\ m-1 < \mu \leq m,\ 0 < t < 1, \tag{11}$$

$$y^{(k)}(0) = 0,\ k = 0, 1, \ldots, m-1. \tag{12}$$

Operating with J^{μ} on both sides of equation (11) and using the initial conditions (12) we obtain,

$$y(t) = J^{\mu}[1] + J^{\mu}[y^2], \tag{13}$$

Using ADM and replace the nonlinear term $f(y) = y^2$ by its corresponding Adomian polynomials we have,

$$y_0 = J^{\mu}[1], \tag{14}$$

$$y_n = J^{\mu}[A_{n-1}],\ n \geq 1. \tag{15}$$

from the two relations (14) and (15), the six-terms approximation are,

$$\phi_6 = \sum_{k=0}^{5} C_k t^{(2k+1)\mu}, \tag{16}$$

where, the coe¢ cients are given by,

$$C_0 = \frac{1}{\Gamma(\mu+1)},\ C_1 = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} C_0^2,$$

$$C_2 = \frac{\Gamma(4\mu+1)}{\Gamma(5\mu+1)}(2C_0C_1),\ C_3 = \frac{\Gamma(6\mu+1)}{\Gamma(7\mu+1)}(2C_0C_2 + C_1^2),$$

$$C_4 = \frac{\Gamma(8\mu+1)}{\Gamma(9\mu+1)}(2C_0C_3 + 2C_1C_2),\ C_5 = \frac{\Gamma(10\mu+1)}{\Gamma(11\mu+1)}(2C_0C_4 + 2C_1C_3 + C_2^2).$$

The solution of the problem (14)-(15) by using the numerical method is:

$$h^{-\mu} \sum_{j=0}^{n} w_j^{(\mu)} - y_n^2 = 1$$

where, $t_n = nh$, $y_n = y(t_n)$, $w_j^{(\mu)} = (-1)$, $\binom{\mu}{j}$, $(n, j = 0, 1, 2, \ldots)$. Therefore, we get

$$h^{-\mu} \sum_{j=0}^{n} (-1)^j \frac{\Gamma(\mu+1)}{\Gamma(j+1)\Gamma(\mu+1)} y_{n-j} - y_n^2 = 1,$$

then,

$$y_n = h^{\mu} + h^{\mu} y_{n-1}^2 - \sum_{j=1}^{n} (-1)^j \frac{\Gamma(\mu+1)}{\Gamma(j+1)\Gamma(\mu-j+1)} y_{n-j},$$

$$m - 1 < \mu \leq m,\ t \in [0, T],\ n = m, m+1, \ldots \tag{17}$$

Figures 1.a-1.f illustrate the comparison between ADM solution ($n = 5$) and the numerical solution ($h = 0.01$). For $\mu = 0.5$, the numerical method gives unbounded solution when $t \in [0, 1]$, see Fig. 1.a, while, ADM gives a bounded solution in the same interval, see Fig. 1.b.

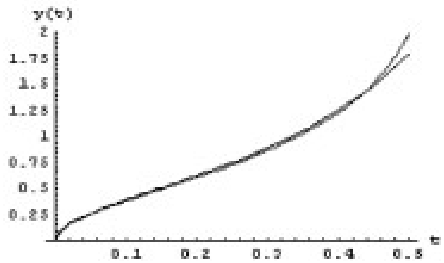

Fig. 1.a: ADM and Numerical Sol. [$\mu = 0.5$]

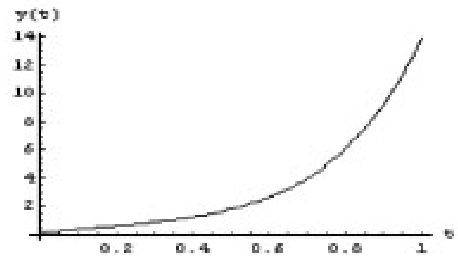

Fig. 1.b: ADM Sol. [$\mu = 0.5$]

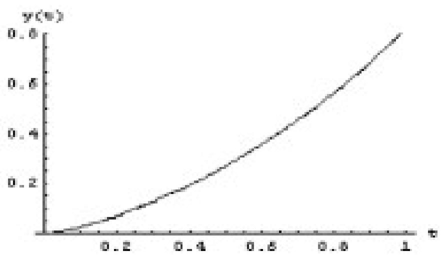

Fig. 1.c: ADM and Numerical Sol. [$\mu = 1.5$]

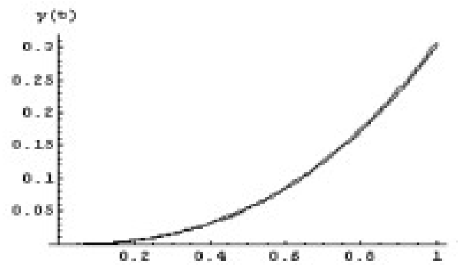

Fig. 1.d: ADM and Numerical Sol. [$\mu = 2.5$]

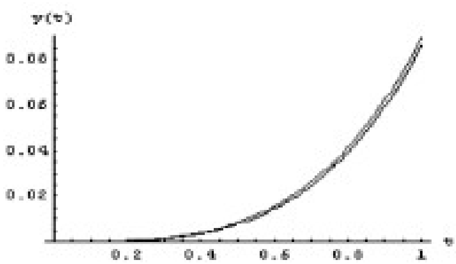

Fig. 1.e: ADM and Numerical Sol. [$\mu = 3.5$]

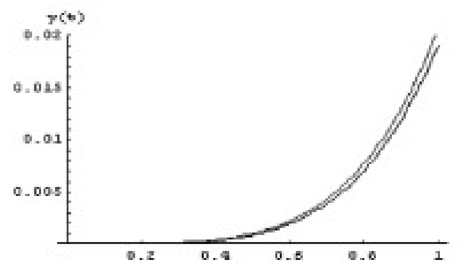

Fig. 1.f: ADM and Numerical Sol. [$\mu = 4.5$]

Table (1) shows the relative error between exact and ADM solution of $\mu = 1$. The value $\mu = 1$ (ODE) is the only case for which we know the exact solution ($y = \tan t$) and our approximate solution is in good agreement with the exact values.

Example 2: Consider the following nonlinear FDE with nonhomogeneous initial conditions,

$$Dy = \frac{9}{4}\sqrt{y} + y, \ t \leq 0, \ 1 < \mu \leq 2, \tag{18}$$
$$y(0) = 1, \ y'(0) = 2.$$

This problem was solved by Nabil Shawagfeh in [20] by using ADM but the given solution was incorrect. Here, we give the correct solution.

Operating with J^{μ} on both sides of (18), we get

Table 1

t	*Exact solution*	*ADM solution*	*Relative error*
0.1	0.100334672	0.100334672	$3.3195594351 \times 10^{-15}$
0.2	0.2027100355	0.2027100355	$1.4756006633 \times 10^{-11}$
0.3	0.3093362496	0.309336249	$1.921474304 \times 10^{-9}$
0.4	0.42279321874	0.42279319296	$6.097058134 \times 10^{-8}$
0.5	0.54630248984	0.54630200191	$8.931495255 \times 10^{-7}$
0.6	0.68413680834	0.68413131533	$8.0291143934 \times 10^{-6}$
0.7	0.84228838046	0.84224495232	0.0000515597
0.8	1.02963855705	1.02937191571	0.000258966
0.9	1.26015821755	1.25879891374	0.001078677
1.0	1.55740772466	1.5513676447	0.003878291

$$y = 1 + 2t + \frac{9}{4} j^{\mu}(\sqrt{y}) + J^{\mu} y. \tag{19}$$

Using ADM and Adomian polynomials to the equation (19) and since the computation of A_n depends heavily on y_0 we will use a slight modification [24]. This will ease the computations considerably. Thus,

$$y_0 = 1, \tag{20}$$

$$y_1 = 2t + \frac{9}{4} j^{\mu}(A_0) + J^{\mu}(y_0), \tag{21}$$

$$y_n = \frac{9}{4} j^{\mu}(A_{n-1}) + J^{\mu}(y_{n-1}),\ n \geq 2. \tag{22}$$

Using relations (20)-(22), the .rst three-terms of the series solution for $\mu = 1.25$ are,

$$y(t) = 1 + 2t + 2.86848t^{1.25} + 1.66715t^{2.25} + 2.0781t^{2.5} + \ldots, \tag{23}$$

and for $\mu = 1.5$,

$$y(t) = 1 + 2t + 2.44482t^{1.5} + 1.27883t^{2.5} + 1.15104t^{3} + \ldots, \tag{24}$$

and for $\mu = 1.75$,

$$y(t) = 1 + 2t + 2.02069t^{1.75} + 0.960889t^{2.75} + 0.593742t^{3.5} + \ldots, \tag{25}$$

and for $\mu = 2$,

$$y(t) = 1 + 2t + \frac{13t^2}{8} + \frac{17}{768} t^3 (32 + 13t) + \cdots. \tag{26}$$

The plots for some values of $\mu \in (1, 2]$ are given in figures 2.a-3.2.d ($n = 5$).

In [20], the mistake is ignoring the third term in equation (21). A comparison between ADM solution and exact solution ($\mu = 2$) is given in Fig. 2.d and its relative error is given in table (2).

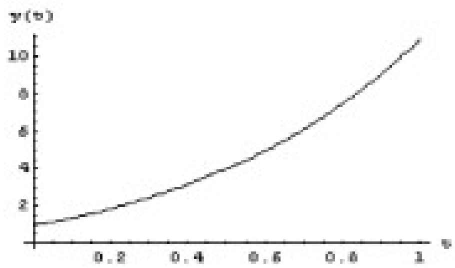

Fig. 2.a: ADM Sol. [$\mu = 1.25$]

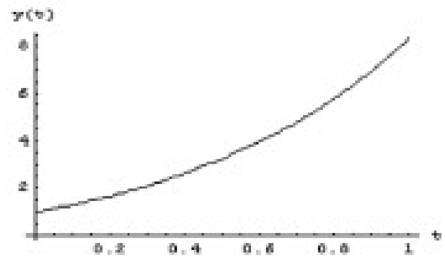

Fig. 2.b: ADM Sol. [$\mu = 1.5$]

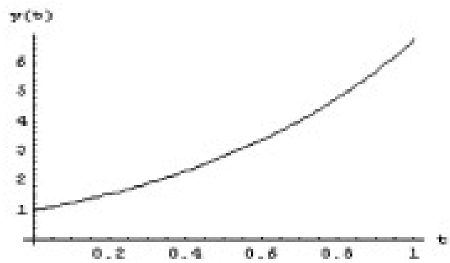

Fig. 2.c: ADM Sol. [$\mu = 1.75$]

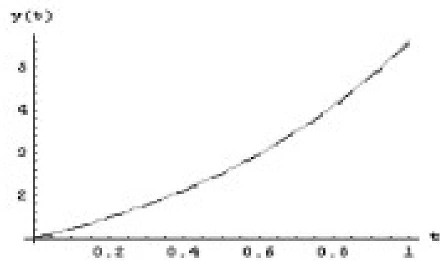

Fig. 2.c: ADM and Exact Sol. [$\mu = 2$]

Table 2

t	*Exact solution*	*ADM solution*	*Relative error*
0.1	1.216978142472	1.216978192217	4.0876110^{-8}
0.2	1.470990374716	1.470993749521	2.2942410^{-6}
0.3	1.767049231411	1.767089933573	0.0000230339715
0.4	2.110740852297	2.11098272054	0.0001145892651
0.5	2.508287449206	2.509262171435	0.0003886006884
0.6	2.966616537584	2.969687894858	0.0010353064631
0.7	3.493437642763	3.501601616183	0.0023369455119
0.8	4.097327267145	4.116482251175	0.0046749948883
0.9	4.787822988191	4.82867126594	0.0085317017462
1.0	5.575527649637	5.656298489041	0.0144866718416

Example 3: Consider the following nonlinear FDE,

$$Dy + D^{1/2}y - 2y^2 = 0,\ y(0) = c. \tag{27}$$

where c is a constant.

This problem was solved by Saha Ray in [25] by using ADM but the given solution was incorrect. Here, we give the correct solution.

Operating with J^1 on both sides of equation (27), we get

$$y = c - J^1\,(D^{1/2}y) + 2J^1\,(y^2). \tag{28}$$

Then we obtain,

$$y = c + \frac{ct^{1/2}}{\Gamma(3/2)} - J^{1/2}(y) + 2J^1(y^2). \tag{29}$$

In [25], the mistake was ignoring the second term in equation (29). Using ADM and the reliable modification [24] to equation (29). Thus,

$$y_0 = c, \tag{30}$$

$$y_1 = \frac{ct^{1/2}}{\Gamma(3/2)} - J^{1/2}(y_0) + 2J^1(A_0) = 2c^2 t, \tag{31}$$

$$y_n = -J^{1/2}\,(y_{n-1}) + 2J^1\,(A_{n-1})\,,\ n \geq 2. \tag{32}$$

From relation (32), we have

$$\begin{aligned} y_2 &= -\frac{8c^2t^{3/2}}{3\sqrt{\pi}} + 4c^2t^2, \\ y_3 &= c^2t^2 - \frac{128c^2t^{5/2}}{15\sqrt{\pi}} + 8c^2t^2, \\ y_4 &= \frac{4}{105}c^2t^{5/2}\left(105c\sqrt{t}(1+4c^2t) - \frac{4(7+152c^2t}{\sqrt{\pi}}\right), \\ &\ \vdots \end{aligned} \tag{33}$$

Therefore, the solution of equation (27) is:

$$y(t) = c + (2c^2t) + \left(-\frac{8c^2t^{3/2}}{3\sqrt{\pi}} + 4c^2t^2\right) + \left(c^2t^2 - \frac{128c^3t^{5/2}}{15\sqrt{\pi}} + 8c^4t^3\right) + \cdots. \tag{34}$$

Now, consider the special case when $c = 1$, equation (27) will be.

$$D_y + D^{1/2}y - 2y^2 = 0,\ y(0) = 1. \tag{35}$$

From equation (34), the solution of equation (35) will be,

$$y(t) = 1 + (2t) + \left(-\frac{8t^{3/2}}{3\sqrt{\pi}} + 4t^2\right) + \left(t^2 - \frac{128t^{5/2}}{15\sqrt{\pi}} + 8t^3\right) + \cdots. \tag{36}$$

Consequently, the practical solution may be taken as n-term approximation to y as follow,

$$\phi_n = \sum_{i=0}^{n-1} y_i. \tag{37}$$

Figures 3.a-3.c show ADM solution of problem (27) at different values of n. Table (3) shows the absolute error between the truncated series at different values of n. We see from this table that, as the number of terms n increases, the error will be decrease.

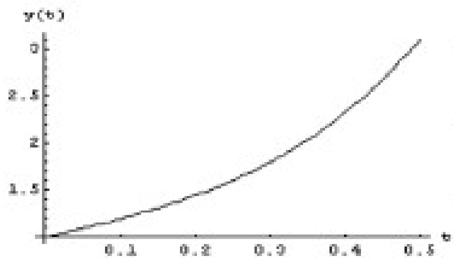

Fig. 3.a: ADM Sol. [n = 5]

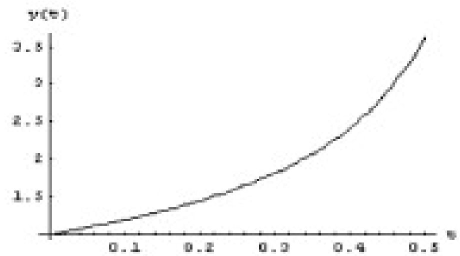

Fig. 3.b: ADM Sol. [n = 10]

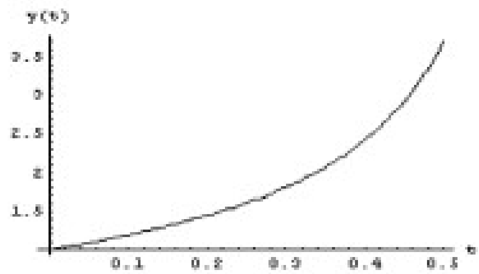

Fig. 3.c: ADM Sol. [n = 15]

Table 3.3

t	$\lvert\phi_5 - \phi_3\rvert$	$\lvert\phi_{10} - \phi_5\rvert$	$\lvert\phi_{15} - \phi_{10}\rvert$	$\lvert\phi_{20} - \phi_{15}\rvert$
0.0	0	0	0	0
0.1	0.00234004	0.000076351	1.99985×10^{-8}	4.93579×10^{-12}
0.2	0.0179593	0.0010773	8.40649×10^{-7}	2.46319×10^{-9}
0.3	0.0833554	0.0130684	0.0000717527	7.04296×10^{-7}
0.4	0.260574	0.0974659	0.00419549	0.000197369
0.5	0.637506	0.507904	0.0890769	0.0161493

Example 4: Consider the nonlinear FDE,

$$D^{1/2}y - y^2 = \Gamma(3/2) - t,\ y(0) = 0, \tag{38}$$

which has the exact solution $t^{1/2}$.

Applying $J^{1/2}$ to both sides of equation (38), we obtain

$$y = t^{1/2} - \frac{t^{3/2}}{\Gamma(5/2)} + J^{1/2}(y^2). \tag{39}$$

Using ADM to the equation (39), we get

$$y_0 = t^{1/2} - \frac{t^{3/2}}{\Gamma(5/2)}, \tag{40}$$

$$y_n = J^{1/2}(A_{n-1}),\ n \geq 1. \tag{41}$$

From the relations (40) and (41), we have

$$y_0 = \sqrt{t} - \frac{4t^{3/2}}{3\sqrt{\pi}},$$

$$y_1 = \frac{4t^{3/2}(105\pi - 224\sqrt{\pi}t + 128t^2)}{315\pi^{3/2}}, \tag{42}$$

$$y_2 = \frac{128t^{5/2}(72765\pi^{3/2} - 216216\pi t + 22580\sqrt{\pi}t^2 - 81920t^3)}{327425\pi^{5/2}}$$

$$\vdots$$

From equation (42), the solution of equation (38) will be,

$$y(t) = \left(\sqrt{t} - \frac{4t^{3/2}}{3\sqrt{\pi}}\right) + \left(\frac{4t^{3/2}(105\pi - 224\sqrt{\pi}t + 128t^2}{315\pi^{3/2}}\right)$$

$$+\left(\frac{128t^{5/2}(72765\pi^{3/2} - 216216\pi t + 22580\sqrt{\pi}t^2 - 81920t^3)}{327425\pi^{5/2}}\right) + \cdots. \tag{43}$$

Consequently, the practical solution may be taken as,

$$\phi_n = \sum_{i=0}^{n-1} y_i. \tag{44}$$

The comparison between ADM and exact solution are given in .gures 4.a-4.b. It is clear that, as the number of terms n increases, the solution will be more accurate.

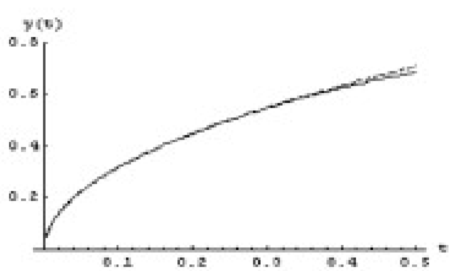

Fig. 4.a: ADM and Exact Sol. [*n* = 5]

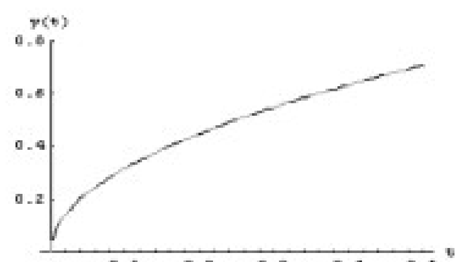

Fig. 4.b: ADM and Exact Sol. [*n* = 10]

Example 5: Consider the nonlinear FDE,

$$D^{3/2}y = \frac{1}{2}y^2 + t^2,\ 0 < t \le 1, \tag{45}$$

$$y(0) = 0,\ y'(0) = 0.$$

Using ADM to the equation (45), we get

$$y_0 = J^{3/2}\,(t^2) \tag{46}$$

$$y_n = \frac{1}{2}J^{3/2}(A_{n-1}),\ n \ge 1. \tag{47}$$

From the relations (46) and (47), the first three-terms of the series solution are,

$$y(t) = \left(\frac{32t^{7/2}}{105\sqrt{\pi}}\right) + \left(\frac{1048576t^{17/2}}{1206079875\sqrt{\pi}}\right) + \left(\frac{140737488355328t^{27/2}}{5778842968659698437 5\sqrt{\pi}}\right) + \cdots. \tag{48}$$

Figures 5.a-5.d show ADM solution of problem (45) at different values of m.

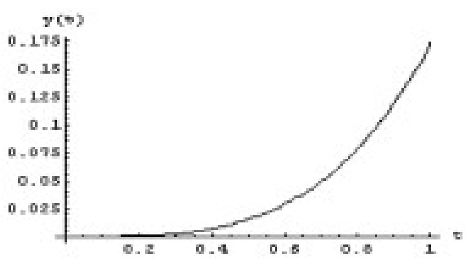

Fig. 5.a: ADM Sol. [m = 5]

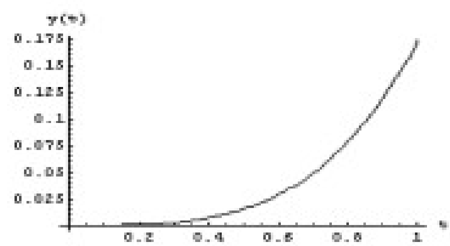

Fig. 5.b: ADM Sol. [m = 10]

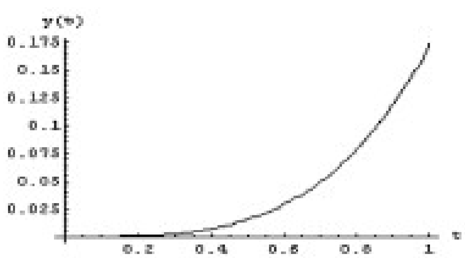

Fig. 5.c: ADM Sol. [m = 15]

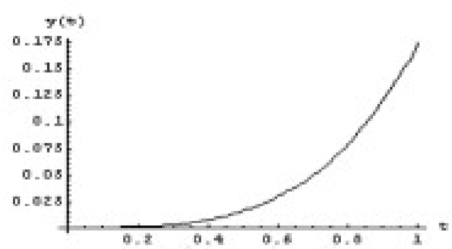

Fig. 5.d: ADM Sol. [m = 20]

Now, we will use Theorem 3 to evaluate the maximum absolute truncated error of the series solution (48). So, we evaluate the following values,

(1) **Lipschitz constant (L):**

$$|\,f(y) - f(z)| = |y^2 - z^2|$$

$$\leq |y+z|\,|y-z|$$
$$\leq 2|y-z| \quad \Rightarrow \ L = 2.$$

(2) M: $|a(\tau)| \leq \frac{1}{2} \quad \Rightarrow M = \frac{1}{2}$

(3) α: $\alpha = \frac{LMT^{\mu}}{\Gamma(\mu+1)} = \frac{1}{\Gamma(5/2)}$.

(4) $\max_{t\in J}|y_1(t)| = \frac{1048576}{1206079875\sqrt{\pi}}$.

(5) The maximum error.

$$\max_{t\in J}\left|y(t) - \sum_{i=0}^{m} y_i(t)\right| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in J}|y_1(t)|$$

(i) For $m = 5$ $\quad \max_{t\in J}\left|y(t) - \sum_{i=0}^{5} y_i(t)\right| \leq 0.00047693.$

(ii) For $m = 10$ $\quad \max_{t\in J}\left|y(t) - \sum_{i=0}^{10} y_i(t)\right| \leq 0.000114889.$

(iii) For $m = 15$ $\quad \max_{t\in J}\left|y(t) - \sum_{i=0}^{15} y_i(t)\right| \leq 0.0000276756.$

(iv) For $m = 20$ $\quad \max_{t\in J}\left|y(t) - \sum_{i=0}^{20} y_i(t)\right| \leq 6.66678 \times 10^{-6}.$

Example 6: Consider the nonlinear FDE,

$$D^{5/2}y = \frac{1}{4}y^4 + t,\ 0 < t \leq 1, \tag{49}$$
$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$$

Using ADM to the equation (49), we get

$$y_0 = J^{5/2}(t), \tag{50}$$
$$y_n = \frac{1}{4}J^{5/2}(A_{n-1}),\ n \geq 1. \tag{51}$$

From the relations (50) and (51), the .rst three-terms of the series solution are,

$$y(t) = \left(\frac{16t^{7/2}}{105\sqrt{\pi}}\right) + \left(\frac{274877906944t^{33/2}}{2009196669692953125\pi^{3/2}}\right)$$

$$+\left(\frac{633825300114114700748351602688t^{59/2}}{581660873631656476634839498138126045898437 5\pi^{5/2}}\right)+\cdots \quad (52)$$

Figures 6.a-6.d show ADM solution of problem (49) at different values of m.

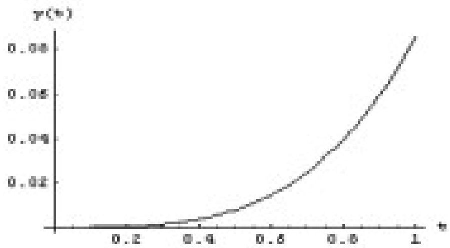

Fig. 6.a: ADM Sol. [m = 5]

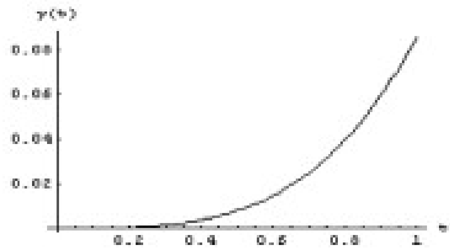

Fig. 6.b: ADM Sol. [m = 10]

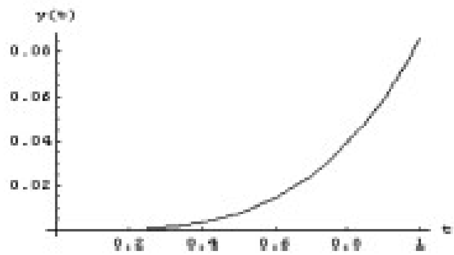

Fig. 6.c: ADM Sol. [m = 15]

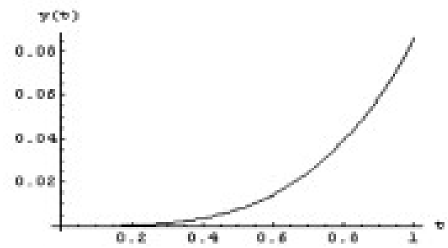

Fig. 6.d: ADM Sol. [m = 20]

Now, we will evaluate the maximum absolute truncated error of the series solution (52). So, we evaluate the following values,

(1) L: $|f(y)-f(z)| = |y^4 - z^4| \le |y^2 + z^2|\,|y+z|\,|y-z|$

$\le 4|y-z| \quad \Rightarrow L = 4.$

(2) M: $|\alpha(\tau)| \le \dfrac{1}{4} \quad \Rightarrow M = \dfrac{1}{4}$

(3) α: $\alpha = \dfrac{LMT^{\mu}}{\Gamma(\mu+1)} = \dfrac{1}{\Gamma(7/2)}.$

(4) $\max_{t\in J}|y_1(t)| = \dfrac{274877906944}{2009196669692953125\pi^{3/2}}.$

(5) **The maximum error:**

(i) For $m = 5$: $\quad \max_{t\in J}\left|y(t)-\sum_{i=0}^{5} y_i(t)\right| \le 8.66908\times 10^{-11}.$

(ii) For $m = 10$: $$\max_{t\in J}\left|y(t)-\sum_{i=0}^{10}y_i(t)\right|\le 2.13841\times 10^{-13}.$$

(iii) For $m = 15$: $$\max_{t\in J}\left|y(t)-\sum_{i=0}^{15}y_i(t)\right|\le 5.27486\times 10^{-16}.$$

(iv) For $m = 20$: $$\max_{t\in J}\left|y(t)-\sum_{i=0}^{20}y_i(t)\right|\le 1.30116\times 10^{-18}.$$

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